

Less Regular Exceptional and Repeating Prime Number Multiplets

H. J. Weber

Department of Physics

University of Virginia

Charlottesville, VA 22904, U.S.A.

January 21, 2013

Abstract

New exceptional (i.e. non-repeating) prime number multiplets are given and formulated in terms of arithmetic progressions, along with laws governing them. Accompanying repeating prime number multiplets are pointed out. Prime number multiplets with less regular distances are studied.

MSC: 11N05, 11N32, 11N80

Keywords: Exceptional, repetitious prime number multiplets.

1 Introduction

In Refs. [1],[2],[3] a variety of prime number multiplets have been discussed, most of which exhibit a regular distance pattern. The reason for restricting attention to them is the enormously complex mix of regularities with chaotic properties of prime numbers.

The well-known exceptional triplet $3, 5, 7$ is the only case of three successive primes in the arithmetic progression $3 + 2n$ for

$n = 0, 1, 2$. All others are composites or single primes, such as 23 for $n = 10$, and 37 for $n = 17$, and ordinary twins like 11, 13 for $n = 4, 5$ and 17, 19 for $n = 7, 8$ etc.

More general exceptional triplets [1], [2] such as $3, 3+2d, 3+4d$ of primes at equal distance $2d$, $(3, d) = 1$ translate into successive prime values in the arithmetic progression $3 + 2dn$. There is at most one prime number triplet in it (for $n = 0, 1, 2$) and all others are composites, single primes or twins at the distance $2d$, but no k -tuple of primes for $k > 3$.

For any odd prime p , in terms of the arithmetic progression $p + 2dn$ with $(d, p) = 1$, there is at most one p -tuple of primes for $n = 0, 1, 2, \dots, p-1$ [2], but no k -tuples for $k > p$. In Section 3 it is shown that there are accompanying $(p-1)$ -tuples of primes that usually repeat.

We also generalize the prime 3 and triplets in (ii) of Theor. 2.3 [1] to an arbitrary odd prime p and p -tuples of primes. This continues the quest for uncovering more and deeper laws governing prime number multiplets.

2 New Exceptional Prime Number Multiplets

Let us start with a few samples of prime number multiplets that generalize the exceptional triplets in (ii) of Theor. 2.3 [1].

Given any distance $2d_1$ not divisible by 5 such that $5 + 2d_1$ is prime, we can extend $5, 5 + 2d_1, 5 + 2d_2, 5 + 2d_3, 5 + 2d_4$ to a maximum-length exceptional quintet of prime numbers using $5|d_j - d_1$. That is to say, there is at most one such quintet for a given distance pattern, and they cannot repeat.

Example 2.1. For $d_1 = 1, 3, 4, 6, 7, \dots$

$$5, 7, 19, 31, 43 \ [2, 2(1+5), 2, 2(1+5)] - 7, -19, -31, -43$$

$$5, 11, 17, 23, 29 \ [6, 6, 6, 6]$$

$$5, 11, 37, 43, 59 \ [6, 2(3+10), 6, 2(3+5)] - 11, -17, -23, -29$$

5, 13, 31, 59, 67 $[8, 2(4 + 5), 2(4 + 10), 8] - 13, -31, -59, -67$
5, 17, 29, 41, 53 $[12, 12, 12, 12] - 7, -19, -31, -43$.

The continuation to negative numbers is given behind the distance pattern in brackets.

None of these quintets (nonets) can be extended to 6 (or 10) primes in a row with the same or similar distance pattern. And they do not repeat, i.e. are all exceptional. The same is the case for 5, 19, 23, 37, 41 at distances $[14, 4, 14, 4]$ with $5|14 - 4$, $5 \nmid 14$.

Similar rules apply to septets for $d_1 = 2, 3, 5, 6, \dots$

7, 11, 29, 47, 79, 83, 101 $[4, 2(2 + 7), 18, 2(2 + 2 \cdot 7), 4, 18]$
3, -29, -47, -79, -83, -101
7, 13, 19, 53, 59, 79, 113 $[6, 6, 2(3 + 2 \cdot 7), 6, 2(3 + 7), 34]$
7, 17, 41, 79, 89, 113, 137 $[10, 2(5 + 7), 2(5 + 2 \cdot 7), 10, 24, 24]$
7, 19, 31, 43, 83, 137, 149 $[12, 12, 12, 2(6 + 2 \cdot 7), 2(6 + 3 \cdot 7), 12]$.

These are all special cases of the following general law.

Theorem 2.2. *Let p be an odd prime, $p|d_j - d_1$, $p \nmid d_1$, $d_j > 0$. Then there is at most one prime number p -tuple $p, p + 2d_1, \dots, p + 2d_{p-1}$ with distance pattern $[2d_1, 2d_2, \dots, 2d_{p-1}]$.*

Proof. If $p, p + 2d_1, \dots, p + 2d_{p-1}$ is a prime number p -tuple, then $p_3 = p_2 + 2d_2 \equiv p_1 + 4d_1 \pmod{p}, \dots, p_{p-1} \equiv p_1 + 2(p-1)d_1 \pmod{p}$. There is no other such p -tuple, because one of p odd numbers in a row at the same distance $2d_1 \pmod{p}$ is divisible by p . \diamond

Corollary 2.3. *There is a p -tuple of primes with a distance pattern $2d_j \pmod{p}$ of Theor. 2.2.*

Proof. With $pn + 2d_1$ forming an arithmetic progression, Dirichlet's theorems allows us picking n_1 so that $pn_1 + 2d_1 = p_1$ is prime. Likewise, we pick n_2 in $pn_2 + 2d_2 = p_2$ so it is prime, etc. The proof in Theor. 2.2 shows that there is no other such p -tuple of primes. \diamond

Using this principle we can construct exceptional p -tuples of primes for any odd prime number p as follows.

Example 2.4. The exceptional septet 7, 13, 47, 67, 73, 79, 113 has the distances $[6, 2(3 + 2 \cdot 7), 2(3 + 7), 6, 6, 34]$, where each distance has the form $2 \cdot 3 + 2 \cdot 7n$ in accord with Theor. 2.2. It cannot be continued to an octet because $113 + 2(7n + 3) = 7(17 + 2n)$ factorizes.

Likewise, the 11-tuple of primes

$$11, 13, 37, 61, 107, 109, 199, 223, 269, 271, 383 \quad (1)$$

with distances

$$[2, 2(1 + 11), 24, 2(1 + 2 \cdot 11), 2, 2(1 + 4 \cdot 11), 24, 46, 2, 2(1 + 5 \cdot 11)] \quad (2)$$

cannot be continued to a 12-tuple, as $383 + 2(11n + 1) = 11(5 \cdot 7 + 2n)$ factorizes.

The exceptional 13-tuple of primes

$$13, 17, 47, 103, 107, 137, 167, 197, 227, 257, 313, 317, 347 \quad (3)$$

with the distance pattern

$$[4, 2(2 + 13), 2(2 + 2 \cdot 13), 4, 30, 30, 30, 30, 30, 56, 4, 30] \quad (4)$$

cannot be continued because $347 + 2(2 + 13n) = 13(27 + 2n)$ factorizes.

The 17-tuple of primes

$$17, 19, 89, 193, 229, 367, 607, 643, 883, 919, 1193, 1229, \\ 1231, 1301, 1303, 1373, 1409 \quad (5)$$

with the distances

$$[2, 2(1 + 2 \cdot 17), 2(1 + 3 \cdot 17), 2(1 + 17), 2(1 + 4 \cdot 17), \\ 2(1 + 7 \cdot 17), 36, 240, 36, 2(1 + 8 \cdot 17), 36, 2, 70, 2, 70, 36] \quad (6)$$

stops because $1409 + 2(1 + 17n) = 17(83 + 2n)$ factorizes. These cases follow the general factorization

$$p + 2d_1(p - 1) + (2d_1 + np) = p(2d_1 + n). \quad (7)$$

In a tour de force, we give the following 43–tuple that goes two steps beyond Euler’s optimal sequence of 41 primes generated by $x(x - 1) + 41$, $x = 0, 1, \dots, 40$:

43, 47, 137, 227, 317, 751, 1013, 1103, 1193, 1283, 1373, 1549,
 1553, 1901, 2593, 2683, 2687, 2777, 2953, 2957, 3391, 3739,
 4001, 4091, 4783, 4787, 4877, 4967, 5573, 5749, 5839,
 5843, 6277, 6367, 7489, 8009, 8443, 8447, 8537, 8627, 8803,
 8807, 9241

with the distance pattern

[4, 2(2 + 43), 2(2 + 43), 2(2 + 43), 2(2 + 5 · 43), 2(2 + 3 · 43),
 2(2 + 43), 2(2 + 43), 2(2 + 43), 2(2 + 43), 2(2 + 2 · 43), 4,
 2(2 + 4 · 43), 2(2 + 8 · 43), 2(2 + 43), 4, 2(2 + 43), 2(2 + 2 · 43), 4,
 2(2 + 5 · 43), 2(2 + 4 · 43), 2(2 + 3 · 43), 2(2 + 43), 2(2 + 8 · 43),
 4, 2(2 + 43), 2(2 + 43), 2(2 + 7 · 43), 2(2 + 2 · 43), 2(2 + 43), 4,
 2(2 + 5 · 43), 2(2 + 43), 2(2 + 13 · 43), 2(2 + 6 · 43), 2(2 + 5 · 43),
 4, 2(2 + 43), 2(2 + 43), 2(2 + 2 · 43), 4, 2(2 + 5 · 43)].

In each step, as outlined in the proof of Theor. 2.5 below, we pick the next possible prime. Yet, sometimes there are gaps of hundreds. In general, it is much easier to search for the first long prime sequence than uncover record setting p –tuples at equal distances [3] of comparable length.

Theorem 2.5 *Given any odd prime p , there are infinitely many $2d_1 > 0$ such that $p + 2d_1$ is prime and $p - 2$ multiples pj_i of p so that the sequence $p, p + 2d_1, p + 4d_1 + 2pj_1, \dots, p + 2(p - 1)d_1 + 2pj_{p-2}$ forms a p –tuple of primes. Each p –tuple of primes is of maximum length and exceptional, i.e. will not repeat.*

Remark 2.6. Since the general multiplet member has the form $p(1 + 2j_k) + 2kd_1$, this result may be viewed as the existence of maximum-length succession of primes in some generalized arithmetic progressions, and the following proof clarifies what is meant by this.

Proof. Given the odd prime p , we pick a d_1 so that $p + 2d_1 = p_1$ is prime. There are infinitely many such values by Dirichlet's theorem, because $p + 2d_1$ is an arithmetic progression with d_1 (and $p \nmid d_1$) running. Next, we pick j_1 in $p + 2d_1 + (2d_1 + 2j_1p) = p_2$ so that p_2 is prime. Again by Dirichlet's theorem there is an infinitude of such values j_1 , because $p(1 + 2j_1) + 4d_1$ is an arithmetic progression. And just as in step 1, each of these choices leads to a complete p -tuple of primes, and so on. In step $p - 1$, we pick j_{p-2} so that $p(1 + 2j_{p-2}) + 2(p - 1)d_1$ is prime. This being an arithmetic progression with j_{p-2} running, it can be done again by Dirichlet's theorem in infinitely many ways. This completes the p -tuple and the proof, because no further step is possible in view of the factorization

$$\begin{aligned} & p(1 + 2j_{p-2}) + 2(p - 1)d_1 + (2d_1 + 2j_{p-1}p) \\ &= p[2d_1 + 1 + 2j_{p-2} + 2j_{p-1}]. \diamond \end{aligned} \quad (8)$$

This construction testifies to the unbelievable variety, richness and complexity of the sequence of ordinary prime numbers.

We could also have walked backward at any step, as is shown in the following examples.

Example 2.7. The quintet

$$5, 7, 19, 11, 13 [2, 2(1 + 5), 2(1 - 5), 2] \quad (9)$$

is stopped by $13 + 2(1 + 5n) = 5(3 + 2n)$;

$$5, 7, 19, 11, 3 [2, 2(1 + 5), 2(1 - 5), 2(1 - 5)] \quad (10)$$

by $3 + 2(1 + 5n) = 5(1 + 2n)$;

$$5, 7, -11, -19, -17 [2, 2(1 - 2 \cdot 5), 2(1 - 5), 2] \quad (11)$$

by $-17 + 2(1 + 5n) = 5(-3 + 2n)$;

$$5, 7, -11, -19, -37 [2, 2(1 - 2 \cdot 5), 2(1 - 5), 2(1 - 2 \cdot 5)], \quad (12)$$

by $-37 + 2(1 + 5n) = 5(-7 + 2n)$. Only walking straight left

$$5, -3, -11, -19, -37 [2(1 - 5), 2(1 - 5), 2(1 - 5), 2(1 - 2 \cdot 5)] \quad (13)$$

yields an optimal nonet upon extending the quintet in Eq. 9.

Corollary 2.8. *By working to the left, some p -tuples of primes may be extended to optimal $(2p - 1)$ -tuples.*

Proof. Essentially the same proof as for Theor. 2.5 works to the left, generating p -tuples involving negative prime numbers. \diamond

Example 2.9. Example 2.1 lists several cases of nonets and a 13-tuple. Equation 1 continues as

$$11, -13, -37, -61, -107, -109, -199, -223, -269, -271, -317(14)$$

with the distance pattern

$$[-2(1 + 11), -24, -24, -2(1 + 2 \cdot 11), -2, -2(1 + 4 \cdot 11), -24, -46, -2, 46]. \quad (15)$$

Induction principle for primes. *Let A_1, \dots, A_n be a finite set of formulas involving a finite number of primes. Let A_1, \dots, A_n be true for the primes p_1, \dots, p_k . If A_1, \dots, A_n are taken to be true for a general set of primes P_1, \dots, P_k and it is shown that there are primes $Q_1 > P_1, \dots, Q_k > P_k$ so that A_1, \dots, A_n are true, then A_1, \dots, A_n hold for an infinitude of prime sets.*

The primes Q_1, \dots, Q_k may be found by Euclid's proof of the infinitude of primes [4],[5] or, if arithmetic progressions are involved, by using Dirichlet's theorem. Cor. 2.3, Theors. 2.5, 3.4 are applications of the prime number induction principle. In all of them A_j are arithmetic progressions.

3 Repeating Prime Patterns

The previous section dealt with non-repeating, or exceptional, p -tuples of primes. If they were incredibly numerous and diverse, repeating patterns are even more so, as we exemplify in this section.

Example 3.1. The exceptional quintet

$$5, 11, 17, 23, 29 [6, 6, 6, 6] \quad (16)$$

is followed by a string of quartets at equal distance 6 :

$$\begin{aligned} &41, 47, 53, 59; 61, 67, 73, 79; 251, 257, 263, 269; \\ &601, 607, 613, 619; 641, 647, 653, 659; \dots \end{aligned} \quad (17)$$

Each quartet is preceded and followed by a multiple of 5, such as 35, 65; 55, 85; 245, 275; Probably there is an infinitude of such quartets but, at fixed equal distance 6, this may be as hard to prove as any twin prime conjecture. In terms of the arithmetic progression $5 + 6n$, there is just one quintet of primes accompanied by a string of quartets of primes, but no k -tuples for $k > 5$.

Similarly, the exceptional septet at equal distance [6] 150

$$7, 157, 307, 457, 607, 757, 907 \quad (18)$$

is accompanied by (probably infinitely many) 6-tuples of primes at equal distance 150 :

$$\begin{aligned} &73, 223, 373, 523, 673, 823; \\ &2467, 2617, 2767, 2917, 3067, 3217; \\ &4637, 4787, 4937, 5087, 5237, 5387; \\ &6079, 6229, 6379, 6529, 6679, 6829; \\ &7717, 7867, 8017, 8167, 8317, 8467; \\ &13163, 13313, 13463, 13613, 13763, 13913; \dots \end{aligned} \quad (19)$$

Again, each 6-tuple is preceded and followed, at the same distance 150, by a multiple of 7, such as $823 + 150 = 7 \cdot 139$, $73 - 150 = -7 \cdot 11$; $2467 - 150 = 7 \cdot 331$, $3217 + 150 = 7 \cdot 481$; $13163 - 150 = 7 \cdot 1859$, $13913 + 150 = 7 \cdot 2009$.

Of course, this also holds for the record setting 11-tuples of primes [3]. The first 11-tuple at equal distance 1536160080 is followed by the 10-tuples at the same distance

$$\begin{aligned} &2009803217, 3545963297, 5082123377, 6618283457, \\ &815443537, 9690603617, 11226763697, 12762923777, \\ &14299083857, 15835243937; \end{aligned}$$

$$\begin{aligned}
&2622695717, 4158855797, 5695015877, 7231175957, \\
&8767336037, 10303496117, 11839656197, 13375816277, \\
&14911976357, 16448136437; \\
&2646083851, 4182243931, 5718404011, 7254564091, \\
&8790724171, 10326884251, 11863044331, 13399204411, \\
&14935364491, 16471524571; \\
&3117107701, 4653267781, 6189427861, 7725587941, \\
&9261748021, 10797908101, 12334068181, 13870228261, \\
&15406388341, 16942548421; \\
&3178320413, 4714480493, 6250640573, 7786800653, \\
&9322960733, 10859120813, 12395280893, 13931440973, \\
&15467601053, 17003761133; \\
&3276952243, 4813112323, 6349272403, 7885432483, \\
&9421592563, 10957752643, 12493912723, 14030072803, \\
&15566232883, 17102392963; \dots
\end{aligned} \tag{20}$$

Again, each decuplet is preceded and followed by a multiple of 11, such as

$$\begin{aligned}
2009803217 - 1536160080 &= 11 \cdot 43058467; \\
15835243937 + 1536160080 &= 11 \cdot 1579218547.
\end{aligned}$$

Remark 3.2. Each exceptional p -tuple of primes at a given distance pattern is accompanied by a (possibly empty, or finite, but usually infinite) set of $(p - 1)$ -tuples of primes that are preceded and followed by multiples of the prime p . This is how $(p - 1)$ -tuples are kept from extending into p -tuples.

Example 3.3. The quintets of Example 2.1 are accompanied by the following repeating quartets with distance patterns shorter by one

$$\begin{aligned}
&17, 29, 31, 43 \ [12, 2, 12] \ 47, 59, 61, 73; \\
&137, 149, 151, 163; \ 167, 179, 181, 193; \dots
\end{aligned}$$

None can be continued to a quintet using any of the distances 2, 12, because $5|17-2, 43+2$; $5|17-12, 43+12$; ... i.e. they are preceded and followed by multiples of 5. The first prime $p_1 \equiv 2 \pmod{5}$, the second prime $p_2 \equiv 4 \pmod{5}$, the third $p_3 \equiv 1 \pmod{5}$ and the 4th $p_4 \equiv 3 \pmod{5}$, which follows from $2, 12 \equiv 2 \pmod{5}$. Likewise,

$$\begin{aligned} &41, 47, 73, 79 [6, 26, 6] 191, 197, 223, 229; \\ &11, 17, 43, 59 [6, 26, 16] 41, 47, 73, 89; \\ &131, 137, 163, 179; 191, 197, 223, 239 \end{aligned}$$

with

$$\begin{aligned} &41, 191, 11, 41, 131, 191 \equiv 1 \pmod{5}, \\ &47, 197, 17, 47, 137, 197 \equiv 2 \pmod{5}, \\ &73, 223, 43, 73, 163, 223 \equiv 3 \pmod{5}, \\ &79, 229, 59, 89, 179, 239 \equiv 4 \pmod{5} \end{aligned}$$

from $6, 26 \equiv 1 \pmod{5}$. Similar rules hold for the first septet in Example 2.1 with the accompanying 6-tuples

$$\begin{aligned} &431, 449, 467, 499, 503, 521 [18, 18, 32, 4, 18] \\ &35081, 35099, 35117, 35149, 35153, 35171; \dots \end{aligned}$$

though there are tremendous gaps between them. Again, they are bracketed by multiples of 7 :

$$\begin{aligned} &431 - 18 = 7 \cdot 59, 521 + 18 = 7 \cdot 77; 431 - 4 = 7 \cdot 61, \\ &521 + 4 = 7 \cdot 75; 431 - 32 = 7 \cdot 57, 521 + 32 = 7 \cdot 79; \\ &35081 - 18 = 7 \cdot 5009, 35171 + 18 = 7 \cdot 5027; \dots \end{aligned}$$

For the distance pattern $[4, 18, 18, 32, 4]$ the 6-tuples start after a large gap

$$\begin{aligned} &50047, 50051, 50069, 50087, 50119, 50123; 197887, \\ &197891, 197909, 197927, 197959, 197963; \dots \end{aligned}$$

For the exceptional 11-tuple of primes in Eq. (2) the first three repeating decuplets of primes are the following:

7989996643, 7989996667, 7989996691, 7989996737, 7989996739,
7989996829, 7989996853, 7989996899, 7989996901, 7989996913;
13291266463, 13291266487, 13291266511, 13291266557,
13291266559, 13291266649, 13291266673, 13291266719,
13291266721, 13291266733;
14024111323, 14024111347, 14024111371, 14024111417,
14024111419, 14024111509, 14024111533, 14024111579,
14024111581, 14024111593

with the first prime of each decuplet $\equiv 2 \pmod{p}$ and the following primes at the same distances as the exceptional p -tuple in Eq. (2). None of the decuplets can be extended to an 11-tuple of primes because, upon adding any distance $\equiv 2 \pmod{p}$ to the last prime of any decuplet yields a multiple of 5 and subtracting $2 \pmod{p}$ from the first prime of each decuplet gives a multiple of 11.

Theorem 3.4. *Let p be an odd prime and $p, \dots, pk_j + 2d_1j, j = 1, \dots, p-1$ be an exceptional p -tuple of primes. Then there are infinitely many $(p-1)$ -tuples of primes at the same distances $2d_1j \pmod{p}, j = 1, 2, \dots, p-1$ as for the exceptional p -tuple of primes.*

Proof. In terms of the arithmetic progressions $2d_1j + k_jp, (d_1, p) = 1$ the 2nd prime of the p -tuple and the first of each $(p-1)$ -tuple are in the arithmetic progression $2d_1 + k_1p, \dots$ the $(j+1)$ th of the p -tuple and j th of the $(p-1)$ -tuple are in the arithmetic progression $2d_1j + k_jp$ for $j = 1, 2, \dots, p-1$. Dirichlet's theorem for arithmetic progressions allows for an infinity of primes in each of these arithmetic progressions. We pick the first prime in each $(p-1)$ -tuple so that they are in increasing order, then the second primes similarly, etc. Then the distances within each $(p-1)$ -tuple are $2d_1j \pmod{p}$ which are the same as for the leading p -tuple. They all start right after a multiple of p at the

distance $2d_1 \pmod{p}$ and end with a multiple of p . \diamond

When the odd prime p in an arithmetic progression $p + 6dn$ is replaced by p^l , $l \geq 2$, then p is no longer available as the first prime of an exceptional p -tuple, leaving only k -tuples with $k \leq p - 1$.

Corollary 3.5. *There is no exceptional p -tuple of primes in the arithmetic progression $p^l + 6dn$, $l \geq 2, p \nmid d$, $n = 0, 1, \dots$*

4 More General Prime Sequences

Example 4.1. The arithmetic progressions $35 + 6n$, $55 + 6n$ contain at most quartets of primes

41, 47, 53, 59; 251, 257, 263, 269; 641, 647, 653, 659; \dots

61, 67, 73, 79; 601, 607, 613, 619; \dots

where multiples of 5 usually precede and end them, such as $59 + 6 = 5 \cdot 13$, $251 - 6 = 5 \cdot 7^2$, $79 + 6 = 5 \cdot 17$. However, the triplet 97, 103, 109 in $55 + 6n$ is preceded by $91 = 7 \cdot 13$, where $5 < 7 < 11$, $13 > 11$. Alternately, the triplet 271, 277, 283 ends before $289 = 17^2$, while $361 = 19^2$ precedes 367, 373, 379. This does not happen for quadruplets. There are no exceptional quintets because $35 = 5 \cdot 7$, $55 = 5 \cdot 11$ are composite.

Theorem 4.2. *Let $p_j | a$, $p_1 < p_2 < \dots < p_A$ be all the odd prime divisors of a in the arithmetic progression $a + 6dn$, $(a, 6d) = 1$. Then multiples of primes $P > p_A$ may eliminate $(p_1 - 1)$ -tuples creating k -tuple fragments of primes starting after a multiple of p_1 and ending before a multiple of P , or starting after a multiple of P and ending before a multiple of p_1 , where $k \leq p_1 - 2$. Prime divisors $p_j > p_1$ may play the same role.*

If $q | d$, $q > 3$ is an odd prime divisor of the distance $6d$ then there are no q -tuples in $a + 6dn$. Let $p' | d$ for all $p' < p_M \leq p_1$. Then $(p_M - 1)$ -tuples of primes are the longest that can occur in $a + 6dn$. If $p_M > p_1$, then $(p_1 - 1)$ -tuples of primes are the longest in $a + 6dn$.

Proof. As shown in Cor. 5 of Ref. [3], a prime divisor $q|d$ is needed to prevent any q -tuple of primes to occur in $a + 6dn$. Any prime divisor $p_j|a$ eliminates exceptional p_j -tuples by its multiples and serves to end k -tuples or precedes a k -tuple by it. Hence there would be $(p_j - 1)$ -tuples of primes if it were not for multiples of $p_k \neq p_j$, $p_k|a$ between two multiples of p_j that eliminate them. As a consequence, there are fragments of $(p_j - 1)$ -tuples of primes, that is smaller m -tuples that start after a multiple of p_j and end before a multiple of p_k . Other fragments start after a multiple of p_{k_1} and end before a multiple of p_{k_2} , or start after a multiple of p_k and end before a multiple of p_j , etc. Example 2.1 shows these cases. Thus, at most k -tuples are allowed with $k \leq p_M - 1$. Thus, the more prime divisors a has the fewer k -tuples of primes will occur in the arithmetic progression $a + 6dn$. The longer the product of successive odd prime divisors the distance $2d$ has starting form 3, the higher the allowed k -tuples are for $q_M < k \leq p_1 - 1 \diamond$.

Corollary 4.3. *Under the conditions of Theor. 4.2, there are infinitely many $(p_M - 1)$ -tuples of primes with equal distances mod p_M starting after some multiple of p_1 and ending ahead of some multiple of p_M . If $p_M > p_1$ then the multipliers are $(p_1 - 1)$ -tuples.*

Proof. This follows along the lines of the proof of Theor. 2.5. \diamond

Note carefully that these multipliers of primes are not necessarily in the arithmetic progression $a + 6dn$.

References

- [1] Weber, H. J., *Regularities of twin, triplet and multiplet prime numbers*, Global J. of Pure and Applied Math. (2011), arXiv:1103.0447 [math.NT].
- [2] Weber, H. J., *Exceptional prime number twins, triplets and multipliers*, preprint arXiv:1102.3075 [math.NT].
- [3] Weber, H. J., *Remarkable and reversible prime number patterns*, preprint submitted April 2011.

- [4] Hardy, G. H., Wright, E. M., *An Introduction to the Theory of Numbers*, Clarendon Press, 5th ed., Oxford (1988).
- [5] Ribenboim, P., *The New Book of Prime Number Records*, Springer, Berlin (1996).
- [6] Dickson, L. E., *History of the Theory of Numbers*, Vol. I, Dover, Mineola, N.Y. (2005), p. 426.